ON THE NÖRLUND MEANS OF VILENKIN-FOURIER SERIES

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Abstract. In this paper we prove and discuss some new (H_p, L_p)-type inequalities of weighted maximal operators of Vilenkin-Nörlund means with non-increasing coefficients. These results are the best possible in a special sense. As applications, both some well-known and new results are pointed out in the theory of strong convergence of Vilenkin-Nörlund means with non-increasing coefficients.

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1. Introduction

The definitions and notations used in this introduction can be found in our next Section. In the one-dimensional case the weak (1,1)-type inequality for maximal operator of Fejér means σ^* can be found in Schipp [25] for Walsh series and in Pál, Simon [24] for bounded Vilenkin series. Fujji [6] and Simon [27] verified that σ^* is bounded from H_1 to L_1. Weisz [40] generalized this result and proved boundedness of σ^* from the martingale space H_p to the Lebesgue space L_p for $p > 1/2$. Simon [26] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. A counterexample for $p = 1/2$ was given by Goginava [13]. Weisz [41] proved that the maximal operator of the Fejér means σ^* is bounded from the Hardy space $H_{1/2}$ to the space $weak - L_{1/2}$. Goginava [17] (see also [31]) proved that weighted maximal operator $\widetilde{\sigma}^*$ is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$. Moreover, the rate of the weights $\left\{\log^2(n+1)\right\}_{n=1}^{\infty}$ in n-th Fejér mean is given exactly. Analogical results for $0 < p < 1/2$ were proved in [32].

Riesz's logarithmic means with respect to Walsh and Vilenkin systems were studied by several authors. We mention, for instance, the papers by Simon [26], Gát, Nagy [11]. In [34] it was proved that the maximal operator of Riesz's means R^* is bounded from the Hardy space $H_{1/2}$ to the space $weak - L_{1/2}$, but is not bounded from the Hardy space H_p to the space L_p, when $0 < p \leq 1/2$. Moreover, there were proved some theorems of boundedness of weighted maximal operators of Riesz's logarithmic means, with respect to Vilenkin-Fourier series.

Móricz and Siddiqi [19] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of L_p function in norm. The case when $q_k = 1/k$ was excluded, since the methods of Móricz and Siddiqi are not applicable to Nörlund logarithmic means. In [9] Gát and Goginava investigated some properties of the Nörlund logarithmic means of functions in the class of continuous functions and in the Lebesgue space L_1. In [33] it was

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proved that there exists a martingale $f \in H_p,\ (0<p\leq 1)$, such that the maximal operator of Nörlund logarithmic means L^* is not bounded in the space L_p. For more information on Nörlund logarithmic means, see paper of Blahota and Gát [3] and Nagy (see [20], [21] and [22]).

In [15] Goginava investigated the behaviour of Cesàro means of Walsh-Fourier series in detail. In the two-dimensional case approximation properties of Nörlund and Cesàro means was considered by Nagy [23]. Weisz [42] proved that the maximal operator $\sigma^{\alpha,*}$ is bounded from the martingale space H_p to the space L_p for $p > 1/(1+\alpha)$. Goginava [14] gave a counterexample, which shows that boundedness does not hold for $0 < p \leq 1/(1+\alpha)$. Simon and Weisz [29] showed that the maximal operator $\sigma^{\alpha,*}$ $(0<\alpha<1)$ of the (C,α) means is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $weak-L_{1/(1+\alpha)}$. In [4] it was also proved that the maximal operator $\widetilde{\sigma}^{\alpha,*}$ is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $L_{1/(1+\alpha)}$. Moreover, this result can not be improved in the following sense:

Theorem BT. *(Blahota, Tephnadze [4]) Let $0<\alpha\leq 1$ and $\varphi:\mathbb{N}_+\to[1,\infty)$ be a non-decreasing function satisfying the condition*
$$\overline{\lim_{n\to\infty}}\frac{\log^{1+\alpha} n}{\varphi(n)}=\infty.$$
Then there exists a martingale $f\in H_{1/(1+\alpha)}(G)$, such that
$$\sup_{n\in\mathbb{N}}\left\|\frac{\sigma_n^{\alpha}f}{\varphi(n)}\right\|_{1/(1+\alpha)}=\infty.$$

It is well-known that Vilenkin systems do not form bases in the space $L_1(G_m)$. Moreover, there is a function in the Hardy space $H_1(G_m)$, such that the partial sums of f are not bounded in L_1-norm. However, in Gát [8] (see also [2]) the following strong convergence result was obtained for all $f\in H_1$:
$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|S_kf-f\|_1}{k}=0.$$

Simon [28] (see also [36]) proved that there exists an absolute constant c_p, depending only on p, such that
$$\frac{1}{\log^{[p]} n}\sum_{k=1}^{n}\frac{\|S_kf\|_p^p}{k^{2-p}}\leq c_p\|f\|_{H_p}^p,\quad (0<p\leq 1) \tag{1}$$
for all $f\in H_p$ and $n\in\mathbb{N}_+$, where $[p]$ denotes the integer part of p. In [37] it was proved that sequence $\{1/k^{2-p}\}_{k=1}^{\infty}$ $(0<p<1)$ in (1) can not be improved.

Weisz considered the norm convergence of Fejér means of Vilenkin-Fourier series and proved the following:

Theorem W1. *(Weisz [39]) Let $p>1/2$ and $f\in H_p$. Then there exists an absolute constant c_p, depending only on p, such that*
$$\|\sigma_kf\|_p\leq c_p\|f\|_{H_p},\ \textit{for all } f\in H_p \textit{ and } k=1,2,\dots.$$

Theorem W1 implies that
$$\frac{1}{n^{2p-1}}\sum_{k=1}^{n}\frac{\|\sigma_kf\|_p^p}{k^{2-2p}}\leq c_p\|f\|_{H_p}^p,\quad (1/2<p<\infty,\ n=1,2,\dots).$$

If Theorem W1 holds for $0<p\leq 1/2$, then we would have

$$\frac{1}{\log^{[1/2+p]}n}\sum_{k=1}^{n}\frac{\|\sigma_k f\|_p^p}{k^{2-2p}}\leq c_p\|f\|_{H_p}^p, \quad (0<p\leq 1/2,\ n=2,3,\dots). \tag{2}$$

However, in [30] it was proved that the assumption $p>1/2$ in Theorem W1 is essential. In particular, we showed that there exists a martingale $f\in H_{1/2}$, such that

$$\sup_n \|\sigma_n f\|_{1/2}=\infty.$$

In [5] it was proved that (2) holds, though Fejér means is not of type (H_p,L_p), for $0<p\leq 1/2$. This result for (C,α) $(0<\alpha<1)$ means when $p=1/(1+\alpha)$ was generalized in [4].

In this paper we prove and discuss some new (H_p,L_p)-type inequalities of weighted maximal operators of Vilenkin-Nörlund means with non-increasing coefficients. These results are the best possible in a special sense. As applications, both some well-known and new results are pointed out in the theory of strong convergence of Vilenkin-Nörlund means.

This paper is organized as follows: in order not to disturb our discussions later on some definitions and notations are presented in Section 2. The main results and some of its consequences can be found in Section 3. For the proofs of the main results we need some auxiliary results of independent interest. Also these results are presented in Section 3. The detailed proofs are given in Section 4.

2. Definitions and Notations

Denote by $\mathbb{N}_+$ the set of the positive integers, $\mathbb{N}:=\mathbb{N}_+\cup\{0\}$. Let $m:=(m_0,m_1,\dots)$ be a sequence of the positive integers not less than 2. Denote by

$$Z_{m_n}:=\{0,1,\dots,m_n-1\}$$

the additive group of integers modulo m_n.

Define the group G_m as the complete direct product of the groups Z_{m_n} with the product of the discrete topologies of Z_{m_n}'s. In this paper we discuss bounded Vilenkin groups, i.e. the case when $\sup_{n\in\mathbb{N}} m_n<\infty$.

The direct product μ of the measures

$$\mu_n(\{j\}):=1/m_n,\ (j\in Z_{m_n})$$

is the Haar measure on G_m with $\mu(G_m)=1$.

The elements of G_m are represented by sequences

$$x:=(x_0,x_1,\dots,x_n,\dots),\ (x_n\in Z_{m_n}).$$

It is easy to give a base for the neighbourhood of G_m :

$$I_0(x):=G_m,\ I_n(x):=\{y\in G_m \mid y_0=x_0,\dots,y_{n-1}=x_{n-1}\}\ (x\in G_m,\ n\in\mathbb{N}).$$

Denote $I_n:=I_n(0)$, for $n\in\mathbb{N}_+$ and

$$e_n:=(0,\dots,x_n=1,0,\dots)\in G_m,\ (n\in\mathbb{N}).$$

It is evident that

$$\overline{I_N}=\left(\bigcup_{k=0}^{N-2}\bigcup_{x_k=1}^{m_k-1}\bigcup_{l=k+1}^{N-1}\bigcup_{x_l=1}^{m_l-1}I_{l+1}\left(x_ke_k+x_le_l\right)\right)\bigcup\left(\bigcup_{k=0}^{N-1}\bigcup_{x_k=1}^{m_k-1}I_N\left(x_ke_k\right)\right). \tag{3}$$

If we define the so-called generalized number system based on m in the following way :

$$M_0:=1,\ M_{n+1}:=m_nM_n \quad (n\in\mathbb{N}),$$

then every $n\in\mathbb{N}$ can be uniquely expressed as

$$n=\sum_{k=0}^{\infty}n_kM_k,\ \text{where } n_k\in Z_{m_k}\ (k\in\mathbb{N}_+)$$

and only a finite number of n_k's differ from zero.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. At first we define the complex-valued function $r_k\left(x\right):G_m\to\mathbb{C}$, the generalized Rademacher functions, by

$$r_k\left(x\right):=\exp\left(2\pi ix_k/m_k\right),\quad \left(i^2=-1,x\in G_m,\ k\in\mathbb{N}\right).$$

Now, define the Vilenkin system $\psi:=(\psi_n:n\in\mathbb{N})$ on G_m as:

$$\psi_n(x):=\prod_{k=0}^{\infty}r_k^{n_k}\left(x\right)\quad \left(n\in\mathbb{N}\right).$$

Specifically, we call this system the Walsh-Paley system, when $m\equiv 2$.

The norm (or quasi-norm) of the space $L_p(G_m)$ $(0<p<\infty)$ is defined by

$$\left\Vert f\right\Vert_p^p:=\int_{G_m}\left\vert f\right\vert^p d\mu.$$

The space $weak-L_p\left(G_m\right)$ consists of all measurable functions f, for which

$$\left\Vert f\right\Vert_{weak-L_p}^p:=\sup_{\lambda>0}\lambda^p\mu\left(f>\lambda\right)<\infty.$$

The Vilenkin system is orthonormal and complete in $L_2\left(G_m\right)$ (see [38]).

Now we introduce analogues of the usual definitions in Fourier-analysis. If $f\in L_1\left(G_m\right)$ we can define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Vilenkin system in the usual manner:

$$\widehat{f}\left(n\right):=\int_{G_m}f\overline{\psi}_nd\mu,\quad S_nf:=\sum_{k=0}^{n-1}\widehat{f}\left(k\right)\psi_k,\quad D_n:=\sum_{k=0}^{n-1}\psi_k,\quad \left(n\in\mathbb{N}_+\right)$$

respectively.

Recall that

$$D_{M_n}\left(x\right)=\left\{\begin{array}{ll}M_n, & \text{if } x\in I_n,\\ 0, & \text{if } x\notin I_n.\end{array}\right. \tag{4}$$

It is also known that (see [1], [10] and [16])

$$D_{sM_n} = D_{M_n} \sum_{k=0}^{s-1} \psi_{kM_n} = D_{M_n} \sum_{k=0}^{s-1} r_n^k, \tag{5}$$

and

$$D_{sM_n - j} = D_{sM_n} - w_{sM_n - 1} \overline{D_j}, \; j = 1, \ldots, M_n - 1. \tag{6}$$

The σ-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by $F_n \, (n \in \mathbb{N})$. Denote by $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ a martingale with respect to $F_n \, (n \in \mathbb{N})$. (for details see e.g. [39]).

The maximal function of a martingale f is defined by

$$f^* := \sup_{n \in \mathbb{N}} \left| f^{(n)} \right|.$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p \; (G_m)$ consist of all martingales, for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ is a martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)} \overline{\psi}_i d\mu.$$

Let $\{q_n : n \geq 0\}$ be a sequence of non-negative numbers. The n-th Nörlund mean is defined by

$$t_n f := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f,$$

where

$$Q_n := \sum_{k=0}^{n-1} q_k.$$

It is well known that

$$t_n f(x) = \int_{G_m} f(t) F_n(x - t) \, dt,$$

where F_n are the so called Nörlund kernels

$$F_n := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} D_k.$$

We always assume that $q_0 > 0$ and $\lim_{n \to \infty} Q_n = \infty$. In this case (see [18]) the summability method generated by $\{q_n : n \geq 0\}$ is regular if and only if

$$\lim_{n \to \infty} \frac{q_{n-1}}{Q_n} = \infty.$$

If $q_n \equiv 1$, then we get the usual n-th Fejér mean and Fejér kernel

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f, \quad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k,$$

respectively.

Let $t,n\in\mathbb{N}$. It is known that (see [7])

$$K_{M_n}(x)=\begin{cases}0, & \text{if } x-x_te_t\notin I_n,\ x\in I_t\backslash I_{t+1},\\ \frac{M_t}{1-r_t(x)}, & \text{if } x-x_te_t\in I_n,\ x\in I_t\backslash I_{t+1},\\ (M_n+1)/2, & \text{if } x\in I_n.\end{cases} \tag{7}$$

The (C,α)-means (Cesàro means) of the Vilenkin-Fourier series are defined by

$$\sigma_n^\alpha f=\frac{1}{A_n^\alpha}\sum_{k=1}^{n}A_{n-k}^{\alpha-1}S_kf,$$

where

$$A_0^\alpha:=0,\quad A_n^\alpha:=\frac{(\alpha+1)\dots(\alpha+n)}{n!},\ \alpha\neq-1,-2,\dots$$

The n-th Riesz logarithmic mean R_n and the Nörlund logarithmic mean L_n are defined by

$$R_nf:=\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_kf}{k},\quad L_nf:=\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_kf}{n-k},$$

respectively, where

$$l_n:=\sum_{k=1}^{n-1}1/k.$$

For the martingale f we consider the following maximal operators:

$$t^*f:=\sup_{n\in\mathbb{N}}|t_nf|,$$

$$\sigma^*f:=\sup_{n\in\mathbb{N}}|\sigma_nf|,\quad \sigma^{\alpha,*}f:=\sup_{n\in\mathbb{N}}|\sigma_n^\alpha f|,$$

$$R^*f:=\sup_{n\in\mathbb{N}}|R_nf|,\quad L^*f:=\sup_{n\in\mathbb{N}}|L_nf|.$$

We also define the following weighted maximal operators:

$$\widetilde{t}^*f:=\sup_{n\in\mathbb{N}}|t_nf|/\log^{1+\alpha}(n+1),$$

$$\widetilde{\sigma}^{\alpha,*}f:=\sup_{n\in\mathbb{N}}|\sigma_n^\alpha f|/\log^{1+\alpha}(n+1),$$

$$\widetilde{\sigma}^*f:=\sup_{n\in\mathbb{N}}|\sigma_nf|/\log^2(n+1).$$

A bounded measurable function a is called a p-atom, if there exists an interval I, such that

$$\int_I a\,d\mu=0,\quad \|a\|_\infty\leq\mu(I)^{-1/p},\quad \operatorname{supp}(a)\subset I.$$

3. Results

Main results and some of its consequences

Theorem 1. *Let $f \in H_{1/(1+\alpha)}$, where $0 < \alpha \leq 1$ and $\{q_n : n \geq 0\}$, be a sequence of non-increasing numbers, such that*

$$n^{\alpha}/Q_n = O(1), \text{ as } n \to \infty, \tag{8}$$

and

$$(q_n - q_{n+1})/n^{\alpha-2} = O(1), \text{ as } n \to \infty. \tag{9}$$

Then there exists an absolute constant c_α, depending only on α, such that

$$\left\| \widetilde{t}^* f \right\|_{1/(1+\alpha)} \leq c_\alpha \|f\|_{H_{1/(1+\alpha)}}.$$

Corollary 1. *(Blahota, Tephnadze [4]) Let $f \in H_{1/(1+\alpha)}$, where $0 < \alpha < 1$. Then there exists an absolute constant c_α, depending only on α, such that*

$$\left\| \widetilde{\sigma}^{\alpha,*} f \right\|_{1/(1+\alpha)} \leq c_\alpha \|f\|_{H_{1/(1+\alpha)}}.$$

Corollary 2. *(Goginava [17], Tephnadze [31]) Let $f \in H_{1/2}$. Then there exists an absolute constant c, such that*

$$\left\| \widetilde{\sigma}^* f \right\|_{1/2} \leq c \|f\|_{H_{1/2}}.$$

Theorem 2. *Let $f \in H_{1/(1+\alpha)}$, where $0 < \alpha < 1$ and $\{q_n : n \geq 0\}$, be a sequence of non-increasing numbers, satisfying condition (8) and (9). Then there exists an absolute constant c_α, depending only on α, such that*

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|t_k f\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)}}{k} \leq c_\alpha \|f\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)}.$$

Corollary 3. *(Blahota, Tephnadze [4]) Let $f \in H_{1/(1+\alpha)}$, where $0 < \alpha < 1$. Then there exists an absolute constant c_α, depending only on α, such that*

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|\sigma_k^{\alpha} f\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)}}{k} \leq c_\alpha \|f\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)}.$$

Corollary 4. *(Blahota, Tephnadze [5], Tephnadze [35]) Let $f \in H_{1/2}$. Then there exists an absolute constant c, such that*

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_{1/2}^{1/2}}{k} \leq c \|f\|_{H_{1/2}}^{1/2}.$$

Remark 1. *For some $\{q_n : n \geq 0\}$ sequences of non-increasing numbers conditions (8) and (9) can be true or false independently.*

Remark 2. *Since Cesàro means satisfy conditions (8) and (9), we immediately obtain from the Theorem BT that the rate of the weights $\left\{\log^{1+\alpha}(n+1)\right\}_{n=1}^{\infty}$ in n-th Nörlund mean can not be improved.*

Some auxiliary results

Weisz proved that the following is true:

Lemma 1. *(Weisz [39]) Suppose that an operator T is σ-linear and for some $0<p\leq 1$*

$$\int\limits_{\overline{I}} |Ta|^p \, d\mu \leq c_p < \infty,$$

for every p-atom a, where I denotes the support of the atom. If T is bounded from L_∞ to L_∞, then

$$\|Tf\|_p \leq c_p \|f\|_{H_p}.$$

We also state three new Lemmas we need for the proofs of our main results but which are also of independent interest:

Lemma 2. *Let $sM_n < r \leq (s+1) M_n$, where $1 \leq s < m_n$. Then*

$$Q_r F_r = Q_r D_{sM_n} - w_{sM_n-1} \sum_{l=1}^{sM_n-2} \left(q_{r-sM_n+l} - q_{r-sM_n+l+1}\right) l\overline{K_l} \tag{10}$$

$$-w_{sM_n-1}\left(sM_n-1\right) q_{r-1}\overline{K_{sM_n-1}} + w_{sM_n} Q_{r-sM_n} F_{r-sM_n}.$$

The next Lemma is generalization of analogical estimation of Cesàro means (see [12])

Lemma 3. *Let $0<\alpha\leq 1$ and $\{q_n : n \geq 0\}$ be a sequence of non-increasing numbers, satisfying conditions (8) and (9). Then*

$$|F_n| \leq \frac{c_\alpha}{n^\alpha}\left\{\sum_{j=0}^{|n|} M_j^\alpha \left|K_{M_j}\right|\right\}.$$

Lemma 4. *Let $0<\alpha\leq 1$ and $\{q_n : n \geq 0\}$ be a sequence of non-increasing numbers, satisfying conditions (8) and (9). If $r \geq M_N$, then*

$$\int_{I_N} |F_r(x-t)| \, d\mu(t) \leq \frac{c_\alpha M_l^\alpha M_k}{r^\alpha M_N}, \ x \in I_{l+1}\left(s_k e_k + s_l e_l\right),$$

where

$$1 \leq s_k \leq m_k - 1, \ 1 \leq s_l \leq m_l - 1 \quad (k = 0, \ldots, N-2, l = k+2, \ldots, N-1)$$

and

$$\int_{I_N} |F_r(x-t)| \, d\mu(t) \leq \frac{c_\alpha M_k}{M_N}, \ x \in I_N\left(s_k e_k\right),$$

where

$$1 \leq s_k \leq m_k - 1, (k = 0, \ldots, N-1).$$

4. Proofs

Proof of Lemma 2. In [16] Goginava proved similar equality for the kernel of Nörlund logarithmic mean L_n. We will use his method.

Let $sM_n < r \leq (s+1) M_n$, where $1 \leq s < m_n$. It is easy to show that

$$\sum_{k=1}^{r} q_{r-k} D_k = \sum_{l=1}^{sM_n} q_{r-l} D_l + \sum_{l=sM_n+1}^{r} q_{r-l} D_l \tag{11}$$
$$:= I + II.$$

By combining (6) and Abel transformation we get that

$$I = \sum_{l=0}^{sM_n-1} q_{r-sM_n+l} D_{sM_n-l} \tag{12}$$
$$= \sum_{l=1}^{sM_n-1} q_{r-sM_n+l} D_{sM_n-l} + q_{r-sM_n} D_{sM_n}$$
$$= D_{sM_n} \sum_{l=0}^{sM_n-1} q_{r-sM_n+l}$$
$$-w_{sM_n-1} \sum_{l=1}^{sM_n-1} q_{r-sM_n+l} \overline{D_l}$$
$$= (Q_r - Q_{r-sM_n}) D_{sM_n}$$
$$-w_{sM_n-1} \sum_{l=1}^{sM_n-2} (q_{r-sM_n+l} - q_{r-sM_n+l+1}) l \overline{K_l}$$
$$-w_{sM_n-1} q_{r-1} (sM_n - 1) \overline{K_{sM_n-1}}.$$

Since
$$D_{j+sM_n} = D_{sM_n} + w_{sM_n} D_j, \quad j = 1, 2, \ldots, sM_n - 1,$$
for II we have that

$$II = \sum_{l=1}^{r-sM_n} q_{r-sM_n-l} D_{l+sM_n} = Q_{r-sM_n} D_{sM_n} + w_{sM_n} Q_{r-sM_n} F_{r-sM_n}. \tag{13}$$

By combining (11)-(13) we obtain (10) and the proof is complete.

Proof of Lemma 3. Let $sM_n < k \leq (s+1) M_n$, where $1 \leq s < m_n$ and sequence $\{q_k : k \geq 0\}$ be non-increasing, satisfying condition

$$\frac{q_0 n}{Q_n} = O(1), \qquad \text{as } n \to \infty. \tag{14}$$

By using Abel transformation we get that

$$Q_n = \sum_{j=1}^{n} q_{n-j} \cdot 1 = \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j + q_0 n \tag{15}$$

and

$$F_n = \frac{1}{Q_n}\left(\sum_{j=1}^{n-1}\left(q_{n-j}-q_{n-j-1}\right)jK_j + q_0 nK_n\right). \tag{16}$$

Since

$$n\left|K_n\right| \le c\sum_{A=0}^{|n|} M_A\left|K_{M_A}\right|, \tag{17}$$

by combining (15) and (16) we immediately get that

$$\begin{aligned}
\left|F_n\right| &\le \frac{c}{Q_n}\left(\sum_{j=1}^{n-1}\left|q_{n-j}-q_{n-j-1}\right| + q_0\right)\sum_{A=0}^{|n|} M_A\left|K_{M_A}\right| \\
&= \frac{c}{Q_n}\left(\sum_{j=1}^{n-1} -\left(q_{n-j}-q_{n-j-1}\right) + q_0\right)\sum_{A=0}^{|n|} M_A\left|K_{M_A}\right| \\
&\le c\frac{2q_0-q_{n-1}}{Q_n}\sum_{A=0}^{|n|} M_A\left|K_{M_A}\right| \\
&\le \frac{c}{Q_n}\sum_{A=0}^{|n|} M_A\left|K_{M_A}\right| \le \frac{c}{n}\sum_{A=0}^{|n|} M_A\left|K_{M_A}\right|.
\end{aligned}$$

Since the case $q_0 n/Q_n = \bar{O}(1)$, as $n\to\infty$, have already been considered, we can exclude it.

Let $0<\alpha<1$. We may assume that $\{q_k : k\ge 0\}$ satisfies conditions (8) and (9) and in addition, satisfies the following

$$\frac{Q_n}{q_0 n} = o(1), \qquad \text{as } \ n\to\infty.$$

It follows that

$$q_n = q_0\frac{q_n n}{q_0 n} \le q_0\frac{Q_n}{q_0 n} = o(1), \qquad \text{as } \ n\to\infty \tag{18}$$

By using (18) we immediately get that

$$q_n = \sum_{l=n}^{\infty}\left(q_l - q_{l+1}\right) \le \sum_{l=n}^{\infty}\frac{1}{l^{2-\alpha}} \le \frac{c}{n^{1-\alpha}} \tag{19}$$

and

$$Q_n = \sum_{l=0}^{n-1} q_l \le \sum_{l=1}^{n}\frac{c}{l^{1-\alpha}} \le cn^{\alpha}. \tag{20}$$

It is easy to show that

$$Q_k\left|D_{sM_n}\right| \le cM_n^{\alpha}\left|D_{sM_n}\right| \tag{21}$$

and

$$\left(sM_n-1\right)q_{k-1}\left|K_{sM_n-1}\right| \le ck^{\alpha-1}M_n\left|K_{sM_n-1}\right| \le cM_n^{\alpha}\left|K_{sM_n-1}\right|. \tag{22}$$

Let

$$n = s_{n_1}M_{n_1} + s_{n_2}M_{n_2} + \cdots + s_{n_r}M_{n_r}, \quad n_1 > n_2 > \cdots > n_r,$$

and

$$n^{(k)} = s_{n_{k+1}}M_{n_{k+1}} + \cdots + s_{n_r}M_{n_r},\ 1 \leq s_{n_l} \leq m_l - 1,\ l = 1, \ldots, r.$$

By combining (21), (22) and Lemma 2 we have that

$$\begin{aligned} &|Q_nF_n| \\ \leq c_\alpha &\left(M_{n_1}^\alpha \left|D_{s_{n_1}M_{n_1}}\right| + \sum_{l=1}^{s_{n_1}M_{n_1}-1} \left|\left(n^{(1)}+l\right)^{\alpha-2}\right| |lK_l| + M_{n_1}^\alpha \left|K_{s_{n_1}M_{n_1}-1}\right| + \left|Q_{n^{(1)}}F_{n^{(1)}}\right| \right). \end{aligned}$$

By repeating this process r-times we get that

$$\begin{aligned} &|Q_nF_n| \\ &\leq c_\alpha \sum_{k=1}^{r} \left(M_{n_k}^\alpha \left|D_{s_{n_k}M_{n_k}}\right| + \sum_{l=1}^{s_{n_k}M_{n_k}-1} \left(n^{(k)}+l\right)^{\alpha-2} |lK_l| + M_{n_k}^\alpha \left|K_{s_{n_k}M_{n_k}-1}\right| \right) \\ &:= I + II + III. \end{aligned}$$

By combining (4), (5) and (7) we obtain that

$$I \leq c_\alpha \sum_{k=1}^{|n|} M_k^\alpha \left|D_{M_k}\right| \leq c_\alpha \sum_{k=1}^{|n|} M_k^\alpha \left|K_{M_k}\right|$$

and

$$\begin{aligned} III &\leq c_\alpha \sum_{k=1}^{r} M_{n_k}^{\alpha-1} \left|M_{n_k}K_{s_{n_k}M_{n_k}} - D_{s_{n_k}M_{n_k}}\right| \\ &\leq c_\alpha \sum_{k=1}^{r} M_k^\alpha \left|K_{M_k}\right|. \end{aligned}$$

Moreover,

$$\begin{aligned} II &= c_\alpha \sum_{k=1}^{r} \sum_{A=1}^{n_k} \sum_{l=s_{A-1}M_{A-1}}^{s_AM_A-1} \left(n^{(k)}+l\right)^{\alpha-2} |lK_l| \\ &= c_\alpha \sum_{k=1}^{r} \sum_{A=1}^{n_{k+1}} \sum_{l=s_{A-1}M_{A-1}}^{s_AM_A-1} \left(n^{(k)}+l\right)^{\alpha-2} |lK_l| \\ &+ c_\alpha \sum_{k=1}^{r} \sum_{A=n_{k+1}+1}^{n_k} \sum_{l=s_{A-1}M_{A-1}}^{s_AM_A-1} \left(n^{(k)}+l\right)^{\alpha-2} |lK_l| \\ &\leq c_\alpha \sum_{k=1}^{r} M_{n_{k+1}}^{\alpha-2} \sum_{A=1}^{n_{k+1}} \sum_{l=s_{A-1}M_{A-1}}^{s_AM_A-1} |lK_l| \\ &+ c_\alpha \sum_{k=1}^{r} \sum_{A=n_{k+1}+1}^{n_k} M_A^{\alpha-2} \sum_{l=s_{A-1}M_{A-1}}^{s_AM_A-1} |lK_l| \\ &:= II_1 + II_2. \end{aligned}$$

By combining (7) and (17) for II_1 we get that

$$II_1 \leq c_\alpha \sum_{k=1}^{r} M_{n_{k+1}}^{\alpha-2} \sum_{A=1}^{n_{k+1}} \sum_{l=s_{A-1}M_{A-1}}^{s_A M_A - 1} \sum_{j=0}^{A} M_j \left|K_{M_j}\right|$$

$$\leq c_\alpha \sum_{k=1}^{n_1} M_k^{\alpha-2} \sum_{A=1}^{k} M_A \sum_{j=0}^{A} M_j \left|K_{M_j}\right|$$

$$\leq c_\alpha \sum_{k=0}^{n_1} M_k^{\alpha-1} \sum_{j=0}^{k} M_j |K_{M_j}|$$

$$= c_\alpha \sum_{j=0}^{n_1} M_j |K_{M_j}| \sum_{k=j}^{n_1} M_k^{\alpha-1}$$

$$\leq c_\alpha \sum_{j=0}^{n_1} M_j^{\alpha} \left|K_{M_j}\right|.$$

By using (17) for II_2 we have similarly that

$$II_2 \leq c_\alpha \sum_{k=1}^{r} \sum_{A=n_{k+1}+1}^{n_k} M_A^{\alpha-1} \sum_{j=0}^{A} M_j \left|K_{M_j}\right|$$

$$\leq c_\alpha \sum_{A=1}^{n_1} M_A^{\alpha-1} \sum_{j=0}^{A} M_j \left|K_{M_j}\right| \leq c_\alpha \sum_{j=0}^{n_1} M_j^{\alpha} \left|K_{M_j}\right|.$$

The proof is complete by combining the estimates above.

Proof of Lemma 4. Let $x \in I_{l+1}(s_k e_k + s_l e_l)$, $1 \leq s_k \leq m_k - 1$, $1 \leq s_l \leq m_l - 1$. Then, by applying (7), we have that

$$K_{M_n}(x) = 0, \text{ when } n > l > k.$$

Suppose that $k < n \leq l$. Moreover, by using (7) we get that

$$|K_{M_n}(x)| \leq c M_k.$$

Let $n \leq k < l$. Then

$$|K_{M_n}(x)| = (M_n + 1)/2 \leq c M_k.$$

If we now apply Lemma 3 we can conclude that

$$Q_r |F_r(x)| \leq c_\alpha \sum_{A=0}^{l} M_A^{\alpha} |K_{M_A}(x)| \tag{23}$$

$$\leq c_\alpha \sum_{A=0}^{l} M_A^{\alpha} M_k \leq c_\alpha M_l^{\alpha} M_k.$$

Let $x \in I_{l+1}(s_k e_k + s_l e_l)$, for some $0 \leq k < l \leq N-1$. Since $x - t \in I_{l+1}(s_k e_k + s_l e_l)$, for $t \in I_N$ and $r \geq M_N$ from (23) we obtain that

$$\int_{I_N} |F_r(x-t)| \, d\mu(t) \leq \frac{c_\alpha M_l^{\alpha} M_k}{r^{\alpha} M_N}. \tag{24}$$

Let $x\in I_N\left(s_ke_k\right),\ k=0,\ldots,N-1$. Then, by applying Lemma 3 and (7) we have that

$$\int_{I_N} Q_r\left|F_r\left(x-t\right)\right|d\mu\left(t\right)\leq\sum_{A=0}^{|r|}M_A^{\alpha}\int_{I_N}\left|K_{M_A}\left(x-t\right)\right|d\mu\left(t\right). \tag{25}$$

Let $x\in I_N\left(s_ke_k\right),\ k=0,\ldots,N-1,\ t\in I_N$ and $x_q\neq t_q$, where $N\leq q\leq|r|-1$. By combining (7) and (25) we get that

$$\int_{I_N} Q_r\left|F_r\left(x-t\right)\right|d\mu\left(t\right)$$

$$\leq c_{\alpha}\sum_{A=0}^{q-1}M_A^{\alpha}\int_{I_N}M_kd\mu\left(t\right)\leq\frac{c_{\alpha}M_kM_q^{\alpha}}{M_N}.$$

Hence,

$$\int_{I_N}\left|F_r\left(x-t\right)\right|d\mu\left(t\right)\leq\frac{c_{\alpha}M_kM_q^{\alpha}}{r^{\alpha}M_N}\leq\frac{c_{\alpha}M_k}{M_N}. \tag{26}$$

Let $x\in I_N\left(s_ke_k\right),\ k=0,\ldots,N-1,\ t\in I_N$ and $x_N=t_N,\ldots,x_{|r|-1}=t_{|r|-1}$. By applying again (7) and (25) we have that

$$\int_{I_N}\left|F_r\left(x-t\right)\right|d\mu\left(t\right)\leq\frac{c_{\alpha}}{r^{\alpha}}\sum_{A=0}^{|r|-1}M_A^{\alpha}\int_{I_N}M_kd\mu\left(t\right)\leq\frac{c_{\alpha}M_k}{M_N}. \tag{27}$$

By combining (24), (26) and (27) we complete the proof of Lemma 4.

Proof of Theorem 1. According to Lemma 1 the proof of the first part of Theorem 1 will be complete, if we show that

$$\int_{\overline{I_N}}\left|\tilde{t}^{*}a(x)\right|^{1/(1+\alpha)}d\mu\left(x\right)<\infty,$$

for every $1/\left(1+\alpha\right)$-atom a. We may assume that a is an arbitrary $1/\left(1+\alpha\right)$-atom with support $I,\ \mu\left(I\right)=M_N^{-1}$ and $I=I_N$. It is easy to see that $t_n\left(a\right)=0$, when $n\leq M_N$. Therefore, we can suppose that $n>M_N$.

Let $x\in I_N$. Since t_n is bounded from L_{∞} to L_{∞} (the boundedness follows from Lemma 3) and $\left\|a\right\|_{\infty}\leq M_N^{1+\alpha}$ we obtain that

$$\left|t_na\left(x\right)\right|\leq\int_{I_N}\left|a\left(t\right)\right|\left|F_n\left(x-t\right)\right|d\mu\left(t\right)$$

$$\leq\left\|a\right\|_{\infty}\int_{I_N}\left|F_n\left(x-t\right)\right|d\mu\left(t\right)$$

$$\leq c_{\alpha}M_N^{1+\alpha}\int_{I_N}\left|F_n\left(x-t\right)\right|d\mu\left(t\right).$$

Let $x\in I_{l+1}\left(s_ke_k+s_le_l\right),\ 0\leq k<l<N$. From Lemma 4 we get that

$$\left|t_na\left(x\right)\right|\leq\frac{c_{\alpha}M_l^{\alpha}M_kM_N^{\alpha}}{n^{\alpha}}. \tag{28}$$

Let $x\in I_N\left(s_ke_k\right),\ 0\leq k<N$. From Lemma 4 we have that

$$\left|t_na\left(x\right)\right|\leq c_{\alpha}M_kM_N^{\alpha}. \tag{29}$$

By combining (3) and (28)-(29) we obtain that

$$\int_{\overline{I_N}}\left|\tilde{t}^{*}a(x)\right|^{1/(1+\alpha)}d\mu\left(x\right)$$

$$=\sum_{k=0}^{N-2}\sum_{s_k=1}^{m_k-1}\sum_{l=k+1}^{N-1}\sum_{s_l=1}^{m_l-1}\int_{I_{l+1}(s_ke_k+s_le_l)}\sup_{n>M_N}\left|\frac{t_na\left(x\right)}{\log^{1+\alpha}(n+1)}\right|^{1/(1+\alpha)}d\mu\left(x\right)$$

$$+\sum_{k=0}^{N-1}\sum_{s_k=1}^{m_k-1}\int_{I_N(s_ke_k)}\sup_{n>M_N}\left|\frac{t_na\left(x\right)}{\log^{1+\alpha}(n+1)}\right|^{1/(1+\alpha)}d\mu\left(x\right)$$

$$\leq\frac{c_\alpha}{N}\sum_{k=0}^{N-2}\sum_{s_k=1}^{m_k-1}\sum_{l=k+1}^{N-1}\sum_{s_l=1}^{m_l-1}\int_{I_{l+1}(s_ke_k+s_le_l)}\sup_{n>M_N}\left|t_na\left(x\right)\right|^{1/(1+\alpha)}d\mu\left(x\right)$$

$$+\frac{c_\alpha}{N}\sum_{k=0}^{N-1}\sum_{s_k=1}^{m_k-1}\int_{I_N(s_ke_k)}\sup_{n>M_N}\left|t_na\left(x\right)\right|^{1/(1+\alpha)}d\mu\left(x\right)$$

$$\leq\frac{c_\alpha}{N}\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{\left(m_k-1\right)\left(m_l-1\right)}{M_{l+1}}\frac{\left(M_l^\alpha M_k\right)^{1/(1+\alpha)}M_N^{\alpha/(1+\alpha)}}{n^{\alpha/(1+\alpha)}}$$

$$+\frac{c_\alpha}{N}\sum_{k=0}^{N-1}\frac{\left(m_k-1\right)}{M_N}M_N^{\alpha/(1+\alpha)}M_k^{1/(1+\alpha)}$$

$$\leq\frac{c_\alpha M_N^{\alpha/(1+\alpha)}}{Nn^{\alpha/(1+\alpha)}}\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{\left(M_l^\alpha M_k\right)^{1/(1+\alpha)}}{M_{l+1}}$$

$$+\frac{c_\alpha}{N}\sum_{k=0}^{N-1}\frac{M_k^{1/(1+\alpha)}}{M_N^{1/(1+\alpha)}}\leq c_\alpha<\infty.$$

The proof is complete.

Proof of Theorem 2. By Lemma 1 the proof of Theorem 2 will be complete, if we show that

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{\left\|t_ka\right\|_{1/(1+\alpha)}^{1/(1+\alpha)}}{k}\leq c_\alpha<\infty,$$

for every $1/\left(1+\alpha\right)$-atom a. Analogously to the proof of Theorem 1 we may assume that a be an arbitrary $1/\left(1+\alpha\right)$-atom with support $I,\ \mu\left(I\right)=M_N^{-1}$ and $I=I_N$ and $n>M_N$.

Let $x\in I_N$. Since t_m is bounded from L_∞ to L_∞ (the boundedness follows from Lemma 3) and $\left\|a\right\|_\infty\leq M_N^{1+\alpha}$, we obtain that

$$\int_{I_N}\left|t_na\left(x\right)\right|^{1/(1+\alpha)}d\mu\leq\left\|a\left(x\right)\right\|_\infty^{1/(1+\alpha)}M_N^{-1}\leq c_\alpha<\infty.$$

Hence

$$\frac{1}{\log n}\sum_{k=M_N}^{n}\frac{\int_{I_N}\left|t_ka\left(x\right)\right|^{1/(1+\alpha)}d\mu}{k}$$

$$\leq \frac{c_\alpha}{\log n}\sum_{k=1}^{n}\frac{1}{k} \leq c_\alpha < \infty.$$

By combining (3) and (28)-(29) we can conclude that

$$\frac{1}{\log n}\sum_{k=M_N+1}^{n}\frac{\int_{\overline{I_N}}\left|t_k a\left(x\right)\right|^{1/(1+\alpha)}d\mu\left(x\right)}{k}$$

$$=\frac{1}{\log n}\sum_{k=M_N+1}^{n}\sum_{r=0}^{N-2}\sum_{s_r=1}^{m_r-1}\sum_{l=r+1}^{N-1}\sum_{s_l=1}^{m_l-1}\frac{\int_{I_{l+1}(s_re_r+s_le_l)}\left|t_k a\left(x\right)\right|^{1/(1+\alpha)}d\mu\left(x\right)}{k}$$

$$+\frac{1}{\log n}\sum_{k=M_N+1}^{n}\sum_{r=0}^{N-1}\sum_{s_r=1}^{m_r-1}\frac{\int_{I_N(s_re_r)}\left|t_k a\left(x\right)\right|^{1/(1+\alpha)}d\mu\left(x\right)}{k}$$

$$\leq\frac{1}{\log n}\left(\sum_{k=M_N+1}^{n}\frac{c_\alpha M_N^{\alpha/(1+\alpha)}}{k^{\alpha/(1+\alpha)+1}}+\sum_{k=M_N+1}^{n}\frac{c_\alpha}{k}\right)<c_\alpha<\infty.$$

The proof is complete.

Proof of Remark 1. Let us see an example. Let

$$q_n := \begin{cases} \frac{1}{\sqrt{n}} & \text{if } n \in \mathbb{N}_+ \\ 0 & \text{if } n = 0. \end{cases}$$

Then sequence is non-increasing and non-negative.

1. Let $0<\alpha<1/2$ arbitrary. It is easy to see, that if $n>2$, then

$$Q_n > \sum_{k=1}^{n-2}\frac{1}{\sqrt{k}} > \int_1^{n-1}\frac{1}{\sqrt{x}}dx = 2\sqrt{n-1}-2$$

Recalling $\alpha<1/2$ we obtain

$$0<\frac{n^\alpha}{Q_n}<\frac{n^\alpha}{2\sqrt{n-1}-2}=O(1)$$

On the other hand

$$\frac{q_n-q_{n+1}}{n^{\alpha-2}}=\frac{1}{\sqrt{1+\frac{1}{n}}\left(\sqrt{1+\frac{1}{n}}+1\right)}n^{\frac{1}{2}-\alpha}\neq O(1).$$

2. Analogously we can show that in the case of $\alpha>1/2$ the situation is the opposite.

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